(1.9) Theorem (De Moivre's Theorem) *.

To show that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \text{ for all integers } n.$$

Proof. When $n = 0$, we have

$$(\cos\theta + i\sin\theta)^0 = 1 = \cos 0 + i\sin 0$$
$$= \cos 0\theta + i\sin 0\theta.$$

When $n = 1$,

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$$
$$= \cos 1\theta + i\sin 1\theta.$$

The result is true for $n = 0$ and $n = 1$.

Now suppose the result is true for $n = k$, i.e.,

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\ &= \cos(k\theta + \theta) + i\sin(k\theta + \theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta.\end{aligned}$$

Thus, the truth of the statement for $n = k$ implies its truth for $n = k + 1$. Hence, by mathematical induction,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \text{ for all } n \in \mathbf{Z}^+.$$

Now suppose n is a negative integer, say $n = -m$, where $m \in \mathbf{Z}^+$. Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m}\\ &= [(\cos\theta + i\sin\theta)^{-1}]^m\\ &= \left[\frac{1}{\cos\theta + i\sin\theta}\right]^m\\ &= \frac{1}{(\cos\theta + i\sin\theta)^m}\end{aligned}$$

* Named after the French mathematician Abraham De Moivre (1667 – 1754).

$$= \frac{1}{\cos m\theta + i \sin m\theta}, \text{ by the result already}$$

$$= \frac{1}{\cos m\theta + i \sin m\theta} \times \frac{\cos m\theta - i \sin m\theta}{\cos m\theta - i \sin m\theta}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i \sin m\theta$$

$$= \cos (-m)\theta + i \sin (-m)\theta$$

$$= \cos n\theta + i \sin n\theta.$$

Thus the theorem is true for all $n \in \mathbf{Z}$.

Example 5. Evaluate $\left(\frac{\sqrt{3} - i}{\sqrt{3} + i}\right)^6$.

Solution. $\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]$

$$(\sqrt{3} - i)^6 = 2^6 \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]^6$$

$$= 2^6 \left[\cos(-\pi) + i \sin(-\pi)\right], \text{ by De Moivre's Th}$$

$$= -2^6.$$

Now $\frac{1}{\sqrt{3} + i} = \frac{1}{\sqrt{3} + i} \times \frac{\sqrt{3} - i}{\sqrt{3} - i}$

$$= \frac{\sqrt{3} - i}{3 + 1} = \frac{1}{4}(\sqrt{3} - i)$$

Therefore, $\frac{1}{(\sqrt{3} + i)^6} = \frac{1}{4^6}(\sqrt{3} - i)^6 = \frac{1}{4^6}(-2^6) = \frac{1}{2^{12}}(-2)^6 = -\frac{1}{2^6}.$

Hence $\left(\frac{\sqrt{3} - i}{\sqrt{3} + i}\right)^6 = \frac{-2^6}{-2^6} = 1.$

Example 6. Prove that

$$(\sin x + i \cos x)^n = \cos n\left(\frac{\pi}{2} - x\right) + i \sin n\left(\frac{\pi}{2} - x\right), \; n \in$$

Solution. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

Hence $$(\sin x + i \cos x)^n = \left[\cos\left(\frac{\pi}{2} - x\right) + i \sin\left(\frac{\pi}{2} - x\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2} - x\right) + i \sin n\left(\frac{\pi}{2} - x\right), \text{ by (1.9)}$$

Example 7. If $x = \cos\theta + i\sin\theta$, show that

$$x^n + \frac{1}{x^n} = 2\cos n\theta \quad \text{and} \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

Solution. Here, $x = \cos\theta + i\sin\theta$.

Therefore, $$x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{1}$$

by (1.9)

$$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

$$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta \tag{2}$$

From (1) and (2), we obtain

$$x^n + \frac{1}{x^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$$

and $$x^n - \frac{1}{x^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = 2i\sin n\theta.$$

(1.10) Applications of De Moivre's Theorem

I. To Express $\cos n\theta$ and $\sin n\theta$ as finite sums of trigonometric functions of θ, where n is a positive integer.

The Binomial Theorem holds for the set of complex numbers. We have

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + \binom{n}{1}\cos^{n-1}\theta\,(i\sin\theta) + \binom{n}{2}\cos^{n-2}\theta\,(i\sin\theta)^2$$

$$+ \binom{n}{3}\cos^{n-3}\theta\,(i\sin\theta)^3 + \binom{n}{4}\cos^{n-4}\theta\,(i\sin\theta)^4$$

$$+ \binom{n}{5}\cos^{n-5}(i\sin\theta)^5 + \cdots$$

(1.Q)

EXERCISE 1.2

Write the following expressions in the form $a + ib$:

(i) $\left(-\sqrt{3}+i\right)^2$ (ii) $(-3i)^4$ (iii) $\left(\dfrac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}\right)^6$

Simplify ;

(i) $\dfrac{(\cos 2\theta + i\sin 2\theta)^5\,(\cos 3\theta - i\sin 3\theta)^6}{(\cos 4\theta - i\sin 4\theta)^7\,(\cos 5\theta + i\sin 5\theta)^8}$

(ii) $\dfrac{(\cos\alpha - i\sin\alpha)^{11}}{(\cos\beta + i\sin\beta)^9}$

(iii) $\dfrac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$

(iv) $\dfrac{\left(3\,\text{cis}\,\frac{\pi}{6}\right)^7}{\left(4\,\text{cis}\,\frac{\pi}{3}\right)^6}$

Prove that

(i) $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

$$= 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\theta+\phi+\pi}{2}\right)$$

Q (ii) $\left(\dfrac{1+\sin x + i\cos x}{1+\sin x - i\cos x}\right)^n = \cos n\left(\dfrac{\pi}{2} - x\right) + i\sin n\left(\dfrac{\pi}{2} - x\right).$

If $2\cos\theta = x + \dfrac{1}{x}$; $2\cos\phi = y + \dfrac{1}{y}$; $2\cos\psi = z + \dfrac{1}{z}$, then prove that

(i) $2\cos(\theta + \phi + \psi) = xyz + \dfrac{1}{xyz}$

(ii) $2\cos(m\theta + n\phi) = x^m y^n + \dfrac{1}{x^m y^n}$.

(i) Find the three cube roots of $8i$

(ii) Find the four 4th roots of each of the complex numbers

$-16i$, 64 and $-2\sqrt{3} + 2i$.